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Dipolar response of an ellipsoidal particle with an anisotropic coating

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Abstract
In this paper we study the response of an ellipsoidal particle with a dielectrically anisotropic coating (the coating dielectric function being different parallel and perpendicular to the coating normal) placed in a constant external electric field. For the coating region we find that potential can be written in terms of solutions to a one-dimensional Heun's equation which is derived from the three-dimensional Gauss equation for the potential in ellipsoidal coordinates. We give solutions to Heun's equation in three forms: for the general case we obtain solutions in terms of a series expansion. For the case of spheroidal particles we write the solutions using hypergeometric functions. For large coating anisotropy we derive a simple form of the solution for the potential. The inside of the ellipsoid and the surroundings are assumed dielectrically isotropic and the potential is therefore given by standard results. By matching the solutions across the boundaries we obtain the ellipsoidal particle polarizability, which is written in terms of the standard depolarization factors and logarithmic derivatives of the Heun's equation solutions. The results above also allow us to obtain the *magnetic* polarizability of a coated ellipsoid in a constant external magnetic field.

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1. Introduction

In many fields of physics shape as well as anisotropy are found to play important roles. For instance fullerenes are found in different shapes and also have electronic properties which are different in different directions. The use of metallic nanoparticles is becoming a hot topic in nanoscale science. Many of their interesting properties stem from the high curvature (i.e., shape) of these small particles. The unique properties of liquid crystals originate from

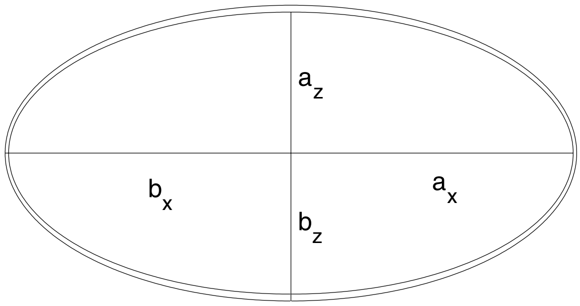

Figure 1. A cut through a coated ellipsoid. The principal semiaxes perpendicular to the paper are of lengths b_y (the outer ellipsoid) and a_y (the inner ellipsoid), with $a_x \geqslant a_y \geqslant a_z$ (and $b_x \geqslant b_y \geqslant b_z$). We assume that a_v and b_v, $v = x, y, z$, are confocal, i.e., related by $b_v^2 = a_v^2 + t$. The ellipsoidal coating has dielectric function $\varepsilon_{c\|}$ in the normal direction and $\varepsilon_{c\perp}$ in the tangential direction. The dielectric functions of the surrounding medium and inner ellipsoid are ε_m and ε_{in} respectively.

its anisotropic structure. Within the field of biological physics the cell is perhaps the most interesting entity. A cell is highly anisotropic and also in general of non-trivial shape.

A convenient way to study matter is to probe the system under investigation using external electric, magnetic or electromagnetic fields. The shape and anisotropy of the system are in general important for the response properties. There have been a number of investigations on how shape and anisotropy affect response properties. In particular we mention: the electric response properties of (coated and uncoated) ellipsoids are described in textbooks [1, 2]. In [3] the polarizability of a sphere with an anisotropic coating is derived. In [4] similarly the electric response of a cylinder with an anisotropic coating is investigated.

In this study we investigate the interaction between a constant external electric or magnetic field and an ellipsoid with a dielectrically anisotropic coating (for instance cells and fullerenes belong to this class of particles) by solving Gauss equation; for the coating region we assume that the dielectric function in the direction parallel to the coating normal is different from the dielectric function in the perpendicular direction. This investigation is organized as follows: in section 2 we turn the relevant three-dimensional Gauss equation for the anisotropic coating region in ellipsoidal coordinates into a one-dimensional Heun's equation. We also recapitulate the standard result for the isotropic case, i.e., for the potential in the inner part of the ellipsoid and its surroundings. Section 3 gives the solutions of Heun's equations for three cases: for the general case the solution is given in terms of a series expansion. For spheroids we derive solutions using hypergeometric functions. For large values of the coating anisotropy a simple asymptotic form is given. In section 4 the solutions inside and outside are matched with the result for the coating region in order to obtain the electric fields as well as the particle polarizability. Finally in section 5 we summarize the results and discuss possible applications.

2. Formulation of the potential problem

In this section we investigate the electric (and magnetic) response of an ellipsoidal particle with an anisotropic coating, where the coating dielectric function is different parallel and perpendicular to the coating normal. In particular, we transform the three-dimensional Gauss equation for the coating region into a one-dimensional Heun's equation.

We consider the coated ellipsoidal particle in a constant external electric field E_0. The lengths of the principal semiaxes of the inner and outer ellipsoids are a_v and b_v ($v = x, y, z$) respectively, where $a_x \geqslant a_y \geqslant a_z$ and $b_x \geqslant b_y \geqslant b_z$ (see figure 1). In a standard fashion

[2] the principal axes are assumed to be confocal, i.e., related as $b_v^2 = a_v^2 + t$. With this choice of the principal axes the solutions in the different regions can be obtained within a single coordinate system, which simplifies the analysis considerably. We denote the isotropic dielectric function of the inner ellipsoid by ε_{in}. The coating has dielectric function $\varepsilon_{\text{c}\parallel}$ in the normal direction and $\varepsilon_{\text{c}\perp}$ in the tangential direction, i.e.,

$$\overleftrightarrow{\varepsilon}_{\text{coat}} = \varepsilon_{\text{c}\parallel}\hat{\xi}\hat{\xi} + \varepsilon_{\text{c}\perp}(\hat{\eta}\hat{\eta} + \hat{\zeta}\hat{\zeta}) \tag{1}$$

where $\hat{\xi}$, $\hat{\eta}$, $\hat{\zeta}$ are unit vectors perpendicular to the ellipsoidal surface ξ = constant, hyperboloid surfaces η = constant and ζ = constant respectively (ξ, η and ζ are ellipsoidal coordinates to be defined below). The isotropic dielectric function of the surrounding medium is denoted by ε_{m}.[3] For the applied potential we write

$$\Phi_{0v} = -E_0 v \tag{2}$$

i.e., the external field E_0 is constant and in the v-direction ($v = x$, y or z). We now seek the induced potential. Inside each of the three regions the potential Φ satisfies Gauss equation given by

$$\vec{\nabla} \cdot (\overleftrightarrow{\varepsilon} \cdot \vec{\nabla}\Phi) = 0. \tag{3}$$

We point out that the case of a permeable ellipsoid with an anisotropic coating in an external homogeneous *magnetic* field B_0 is described by equation (3) as well provided that Φ is taken as the *magnetic scalar potential* and we replace $\overleftrightarrow{\varepsilon} \to \overleftrightarrow{\mu}$, where $\overleftrightarrow{\mu}$ is the magnetic permeability of the different regions [5]. It is natural to use ellipsoidal coordinates [1] when treating the problem above. For the isotropic case (inner part of the ellipsoid and the surrounding medium) $\overleftrightarrow{\varepsilon}$ is constant and scalar and the equation above reduces to Laplace equation, which has standard solutions in ellipsoidal coordinates [1, 2]. For the anisotropic case (coating region) no such standard solutions exist and the rest of this section is dedicated to transforming Gauss equation for the coating region into the known Heun equation [6].

Let us now show that the three-dimensional Gauss equation (3) (together with the coating dielectric function (1)) can be simplified by a proper ansatz for the potential. Let us first recapitulate the definitions of the ellipsoidal coordinates, which are denoted by $p = \xi$, η and ζ, and are given by the roots to the cubic equation (x, y and z are the Cartesian coordinates) [1]

$$\frac{x^2}{a_x^2 + p} + \frac{y^2}{a_y^2 + p} + \frac{z^2}{a_z^2 + p} = 1. \tag{4}$$

For $a_x \geqslant a_y \geqslant a_z$ as we assume here (see figure 1), the ranges of the ellipsoidal coordinates are $\xi \geqslant -a_z^2$, $-a_z^2 \geqslant \eta \geqslant -a_y^2$ and $-a_y^2 \geqslant \zeta \geqslant -a_x^2$. Surfaces of constant ξ are ellipsoids, whereas surfaces of constant η and ζ are hyperboloids of one and two sheets respectively. In particular $\xi = 0$ corresponds to the inner ellipsoidal surface, whereas $\xi = t$ corresponds to the outer surface. The Cartesian coordinates x, y and z can be explicitly expressed in terms of ξ, η and ζ (see for instance [1]). In ellipsoidal coordinates Gauss equation (3), together with the anisotropic dielectric function (1), is

$$\begin{aligned}(\eta - \zeta)R(\xi)\frac{\partial}{\partial\xi}\left[R(\xi)\frac{\partial\Phi}{\partial\xi}\right] + \frac{\varepsilon_{\text{c}\perp}}{\varepsilon_{\text{c}\parallel}}(\zeta - \xi)R(\eta)\frac{\partial}{\partial\eta}\left[R(\eta)\frac{\partial\Phi}{\partial\eta}\right]\\ + \frac{\varepsilon_{\text{c}\perp}}{\varepsilon_{\text{c}\parallel}}(\xi - \eta)R(\zeta)\frac{\partial}{\partial\zeta}\left[R(\zeta)\frac{\partial\Phi}{\partial\zeta}\right] = 0\end{aligned} \tag{5}$$

[3] All four dielectric functions ε_{in}, ε_{m}, $\varepsilon_{\text{c}\parallel}$ and $\varepsilon_{\text{c}\perp}$ are assumed to be spatially independent. However the dielectric functions may in general depend on frequency.

where $R(p) = \left[\left(a_x^2 + p\right)\left(a_y^2 + p\right)\left(a_z^2 + p\right)\right]^{1/2}$, for $p = \xi, \eta, \zeta$. Note that in Gauss equation (5) the dielectric constants $\epsilon_{c\parallel}$ and $\epsilon_{c\perp}$ enter only through the ratio $\epsilon_{c\perp}/\epsilon_{c\parallel}$ which is controlled by the anisotropy of the coating (whereas for isotropic media the dielectric functions appear only through the boundary conditions). We proceed by trying the ansatz ($v = x, y, z$) for the potential (for the external electric field E_0 along the v-direction)

$$\Phi \propto v G_v(\xi). \tag{6}$$

Inserting this ansatz into equation (5) we find that $G_v(\xi)$ can indeed be taken as independent of η and ζ and it satisfies the second-order differential equation

$$G_v''(\xi) + k_v(\xi) G_v'(\xi) + \frac{1}{4}\left[1 - \frac{\varepsilon_{c\perp}}{\varepsilon_{c\parallel}}\right] m_v(\xi) G_v(\xi) = 0 \tag{7}$$

where a prime denotes derivative with respect to the argument, and

$$\begin{aligned} k_v(\xi) &= \frac{\mathrm{d}}{\mathrm{d}\xi} \ln\left[R(\xi)\left(a_v^2 + \xi\right)\right] \\ m_v(\xi) &= \frac{a_x^2 + a_y^2 + a_z^2 - a_v^2 + 2\xi}{R^2(\xi)}. \end{aligned} \tag{8}$$

The ansatz equation (6) for the potential thus turns the three-dimensional Gauss equation (3) for the coating region into a one-dimensional equation (7), where the only variable appearing is the 'radial' ellipsoidal coordinate ξ. We point out that equation (7) applies to all cases $v = x$, y and z, i.e., the external field being along any of the three ellipsoid principal axes.

Let us now show that equation (7) can be turned into the more familiar Heun equation. It is then convenient not to work directly in terms of ξ but rather introduce the variable substitution

$$q \equiv e_i^2 \big/ \left(1 + \xi/a_x^2\right) \tag{9}$$

where we have defined the ellipticity of the inner surface

$$e_i^2 \equiv 1 - a_z^2 \big/ a_x^2. \tag{10}$$

Since, $a_x \geqslant a_y \geqslant a_z$, and $\xi \geqslant -a_z^2$, we have $0 \leqslant e_i^2 \leqslant 1$ and $0 \leqslant q \leqslant 1$. For a spherical surface ($a_x = a_y = a_z$) $e_i^2 = 0$. The surface of the inner ellipsoid corresponds to $q = e_i^2$ ($\xi = 0$) and the surface of the outer ellipsoid is $q = e_o^2$ ($\xi = t$), where we have defined the ellipticity of the outer surface

$$e_o^2 \equiv 1 - \frac{b_z^2}{b_x^2}. \tag{11}$$

Note that $e_i^2 \neq e_o^2$ in the general case. For a thin coating the ellipticities are related: $e_i^2 \approx e_o^2(1 + \bar{\delta})$, where we have defined the relative coating thickness $\bar{\delta} \equiv t \big/ b_x^2$. We also define

$$s \equiv \left(a_x^2 - a_y^2\right) \big/ \left(a_x^2 - a_z^2\right) = \left(1 - a_y^2 \big/ a_x^2\right) \big/ e_i^2. \tag{12}$$

We then have $0 \leqslant s \leqslant 1$; $s = 0$ for an oblate spheroid ($a_x = a_y$), while $s = 1$ for a prolate spheroid ($a_y = a_z$). A coated cylinder ($a_y = a_z$ and $a_x \to \infty$) corresponds to $s = 1$ and $e_i^2 = e_o^2 = 1$. One would also in general define the shape factor $s_o \equiv \left(b_x^2 - b_y^2\right) \big/ \left(b_x^2 - b_z^2\right)$ for the outer ellipsoid. However, using $b_v^2 = a_v^2 + t$, we have that $s = s_o$. The shape of a coated ellipsoid is therefore defined through only three parameters, e_i^2, e_o^2 and s (all in the range [0, 1]). We proceed by writing (see equation (7))

$$G_v(\xi) = G_v(u, s; q) = q^{(1-u)/2} H_v(u, s; q) \tag{13}$$

Table 1. Parameters in Heun's equation in terms of $u = 1/2\ (-1 \pm [1 + 8\varepsilon_{c\perp}\varepsilon_{c\parallel}^{-1}]^{1/2})$, where $\varepsilon_{c\parallel}$ ($\varepsilon_{c\perp}$) is the dielectric function in the direction parallell (perpendicular) to the ellipsoid coating normal. The shape parameter s is defined in equation (12).

v	x	y	z
α_v	$-u/2$	$1-u/2$	$1-u/2$
β	$(1-u)/2$	$(1-u)/2$	$(1-u)/2$
γ	$1/2-u$	$1/2-u$	$1/2-u$
δ_v	$1/2$	$1/2$	$3/2$
ϵ_v	$1/2$	$3/2$	$1/2$
λ_v	$(u-1)u(s+1)/(8s)$	$(u-1)[2s(u-1)+u]/(8s)$	$(u-1)[2(u-1)+su]/(8s)$

where $v = x, y$ or z as before and we choose u to satisfy the indicial equation $(u-1)(u/2+1) + \left(1 - \varepsilon_{c\perp}\varepsilon_{c\parallel}^{-1}\right) = 0$, or equivalently

$$u = u_\pm = -\tfrac{1}{2} \pm \tfrac{1}{2}\left[1 + 8\varepsilon_{c\perp}\varepsilon_{c\parallel}^{-1}\right]^{1/2}. \tag{14}$$

We note here that for an isotropic coating $\varepsilon_{c\perp} = \varepsilon_{c\parallel}$ and we get $u_+ = 1$ and $u_- = -2$. Equation (7) now yields

$$\begin{aligned} &H_v''(q) + \mathcal{L}_v(q)H_v'(q) + \mathcal{M}_v(q)H_v(q) = 0 \\ &\mathcal{L}_v(q) = \frac{\gamma}{q} + \frac{\delta_v}{q-1} + \frac{\epsilon_v}{q-a} \\ &\mathcal{M}_v(q) = \frac{\alpha_v\beta q - \lambda_v}{q(q-1)(q-a)} \end{aligned} \tag{15}$$

where $a = 1/s$, $H_v(q) \equiv H_v(u, s; q)$ and the explicit expressions for the parameters in terms of u are as given in compact form, for all cases, $v = x$, y and z, in table 1. Equation (15) has the form of Heun's equation [6, 7], which is a second-order homogeneous differential equation with *four* regular singular points (at $q = 0, 1, a = 1/s$ and ∞). The parameters must satisfy the relation $\alpha_v + \beta + 1 = \gamma + \delta_v + \epsilon_v$, which is satisfied in our case for the parameters listed in table 1. In a standard fashion we choose the 'normalization' such that $H_v(u, s; q = 0) = 1$. Explicit solutions of Heun's equation will be constructed in the next section. If one solution to Heun's equation is $H_v(u, s; q) \equiv H_v(\alpha_v, \beta, \gamma, \delta_v, \epsilon_v, \lambda_v; q)$, then the second linearly independent solution is $\tilde{H}_v(u, s; q) \equiv q^{1-\gamma} H_v(\alpha_v + 1 - \gamma, \beta + 1 - \gamma, 2 - \gamma, \delta_v, \epsilon_v; \lambda_v', q)$, where $\lambda_v' = \lambda_v + (1-\gamma)(a\delta_v + \epsilon_v)$ [6]. Since u takes two values, at first, there seem to be four solutions for $G_v(u, s; q)$, namely

$$\begin{aligned} G_{1v} &= q^{(1-u_-)/2} H_v(u_-, s; q) \qquad & G_{2v} &= q^{(1-u_+)/2} H_v(u_+, s; q) \\ G_{3v} &= q^{(1-u_-)/2} \tilde{H}_v(u_-, s; q) \qquad & G_{4v} &= q^{(1-u_+)/2} \tilde{H}_v(u_+, s; q). \end{aligned}$$

However, using the fact that $u_+ + u_- = -1$ it is straightforward to show that $G_{1v} = G_{4v}$ and $G_{2v} = G_{3v}$ and there are only two linearly independent solutions as it should. In practice it therefore suffices to find one solution of Heun's equation, and the second solution is obtained by replacing $u_+ \leftrightarrow u_-$ in that solution. In the general case the solution for the potential Φ for the coating region is given by equations (6), (13) and the solution to Heun's equation (15).

The region inside the ellipsoid as well as the surroundings are assumed dielectrically isotropic in this study. For these regions the potential is given by the solutions to Laplace equation (obtained by letting $\varepsilon_{c\perp} = \varepsilon_{c\parallel}$ in equation (7)), which has the standard solutions in ellipsoidal coordinates [1, 2] $G_{1v}(\xi) =$ constant and $G_{2v}(\xi) \propto \int_\xi^\infty \mathrm{d}\xi' / \left[\left(a_v^2 + \xi'\right) R(\xi')\right]$. Converting to the variable q, one finds that $G_{1v}(\xi)$ corresponds to (see equations (9) and (13))

$$H_v(u_+ = 1, s; q) = 1 \tag{16}$$

and $G_{2v}(\xi)$ corresponds to $H_v(u_- = -2, s; q)$ with

$$H_v(u_- = -2, s; q) = \frac{3}{2} \int_0^1 \frac{t^{1/2}}{(1-qt)^{\delta_v}(1-qst)^{\epsilon_v}} \,\mathrm{d}t \tag{17}$$

where the normalization $H_v(u, s; q = 0) = 1$ is maintained. It is interesting to note here that while in general no integral representation for the solution of Heun's equation is known, the above expression (17) gives an integral representation although for a very specific set of values of the parameters (which can be obtained by setting $u = -2$ in table 1) in Heun's equation. In a standard fashion we now introduce the depolarization factors [1, 2] which in terms of the variable q are

$$\begin{aligned} n_v(s; q) &\equiv \frac{1}{3}[f_x f_y f_z]^{1/2} H_v(u = -2, s; q) \\ &= \frac{1}{2}[f_x f_y f_z]^{1/2} \int_0^1 \frac{t^{1/2}}{(1-qt)^{\delta_v}(1-qst)^{\epsilon_v}} \,\mathrm{d}t \end{aligned} \tag{18}$$

where $f_v = f_v(s; q) \equiv 1 - l_v q$ with $l_x = 0, l_y = s$ and $l_z = 1$. The parameters δ_v and ϵ_v are given in table 1. The depolarization factors are dimensionless shape functions and satisfy the sum rule $n_x(s; q) + n_y(s; q) + n_z(s; q) = 1$. From this sum rule one finds that (by symmetry) $n_x = n_y = n_z = 1/3$ for a sphere. It is also straightforward to show that for a cylindrical surface we have $n_x = 0$ and $n_y = n_z = 1/2$ [1].

Let us finally introduce some quantities involving the logarithmic derivatives of the Heun equation solutions, which will be needed while applying the boundary conditions that the potential and the normal component of the displacement field are continuous in section 4. We define

$$\begin{aligned} r_v(u, s; q) &\equiv 1 - f_v(s; q) \left\{ 2q \frac{\partial}{\partial q} \ln[G_v(u, s; q)] \right\} \Bigg|_{q \neq 1} \\ &= 1 - f_v(s; q) \left\{ 1 - u + 2q \frac{\partial}{\partial q} \ln[H_v(u, s; q)] \right\} \Bigg|_{q \neq 1} \end{aligned} \tag{19}$$

where $f_v(s; q)$ has been defined before and we require $q \neq 1$ for reasons to be explained in section 4. Using equations (16)–(18) we find that for the isotropic case we have

$$r_v(u = 1, s; q) = 1 \qquad r_v(u = -2, s; q) = 1 - \frac{1}{n_v(s; q)}. \tag{20}$$

The solution $H_v(u, s; q)$ as given by equation (15), together with the depolarization factors $n_v(s; q)$ (see equation (18)) and $r_v(u, s; q)$ defined in equation (19), completely determines the electric response (and in particular the polarizability) of an ellipsoidal particle with an anisotropic coating, as we will see in section 4. We will in the following section derive expressions for these entities.

3. Solutions of Heun's equation

In this section we present the solution $H_v(u, s; q)$ of Heun's equation and the entity $r_v(u, s; q)$ introduced in the previous section for three cases: for the general case these entities are given in terms of a series expansion. For spheroids (two of the ellipsoid principal axes are equal) we derive expression for $H_v(u, s; q)$ and $r_v(u, s; q)$ in terms of hypergeometric functions. For large $|u|$ a simple asymptotic form is given.

3.1. General case

In this subsection we present the general solution $H_v(u, s; q)$ of Heun's equation and for $r_v(u, s; q)$ in terms of a series expansion. We also investigate the spherical limit.

The solution of Heun's equation (15) can be generated by series expansions around the singular points. We here use the series expansion around $q = 0$ (valid for $|q| < \min(1, a = 1/s) = 1$ [6]) which is

$$H_v(u, s; q) = \sum_{m=0}^{\infty} k_m(u, s) q^m \tag{21}$$

where the coefficients satisfy the recurrence relation

$$(m + 1)(m + \gamma)k_{m+1} - \{[(1 + s)(m - 1 + \gamma) + \delta_v + s\epsilon_v]m + s\lambda_v\} k_m + s(m - 1 + \alpha_v)(m - 1 + \beta)k_{m-1} = 0 \tag{22}$$

with $k_0 = 1$ and $k_{-1} = 0$ and we have left the v-dependence of the expansion coefficients k_m implicit. The parameters are given in table 1. The above series is not defined for $\gamma = -m$ where m is a positive integer ($m = 0, 1, 2, \ldots$). Explicitly we find that the first few coefficients are

$$k_1 = \frac{s\lambda_v}{\gamma} \qquad k_2 = \frac{1}{2(1 + \gamma)} \left\{ [s(\gamma + \epsilon_v) + s\lambda_v + \gamma + \delta_v] \frac{s\lambda_v}{\gamma} - s\alpha_v\beta \right\}. \tag{23}$$

Note that k_1 only depends on s, λ_v and γ, whereas k_2 depends on all parameters appearing in Heun's equation. Note also that with $k_0 = 1$, the normalization of $H_v(u, s; q)$ is such that $H_v(u, s; q = 0) = 1$ as previously. The function $r_v(u, s; q)$ is given by equation (19); to second-order in a power series expansion

$$r_v(u, s; q) \approx u + [(1 - u)l_v - 2k_1]q - 2\big(2k_2 - k_1^2 - l_v k_1\big)q^2 \tag{24}$$

where $l_x = 0, l_y = s$ and $l_z = 1$ as before and k_1 and k_2 are given by equation (23). For the general case $H_v(u, s; q)$ and $r_v(u, s; q)$ can straightforwardly be generated on a computer using the recurrence relation (22). When only results valid up to second-order in q are needed equations (21), (23) and (24) can be used. For the case when $|u| \gg 1$ the limiting forms of $H_v(u, s; q)$ and $r_v(u, s; q)$ given in subsection 3.3 are useful.

When $q \to 1$ (recall that $0 \leqslant q \leqslant 1$) the series expansion above is not valid. The results above for $H_v(u, s; q)$ (the entity $r_v(u, s; q)$ is not defined for $q = 1$) must then be analytically continued to $q = 1$, the so-called two-point connection problem [8]; this problem is difficult and in general no closed form analytic results exist. We therefore leave this problem for future studies. However, for the special case of spheroids the solutions can be written in terms of hypergeometric functions as we will see in the next subsection. The hypergeometric functions have standard analytic continuations to $q = 1$.

Let us finally consider the spherical limit $q \to 0$ while $e_i^2 \to 0$ [1, 2, 9]. From the definition of the ellipsoidal coordinates, equation (4), we find that the spherical radial coordinate $r \equiv \sqrt{x^2 + y^2 + z^2}$ in this limit is related to q (see equation (9)) according to $r^2 = a^2 e_i^2 \big/ q$, where $a = a_x = a_y = a_z$ is the inner radius of the sphere. Using equation (6) together with equation (13) and the fact that $H_v(u, s; q \to 0) = 1$ we find that the potential in the coating region for a sphere with an anisotropic coating is

$$\Phi|_{\text{sphere}} \propto v r^{u-1} \tag{25}$$

where $v = x, y$ or z and $u = u_\pm$ is given in equation (14). This result for the potential agrees with the result obtained in [3] as it should. The result for $r_v(u, s; q)$ in the spherical limit is

$$r_v(u, s; q)|_{\text{sphere}} = u \tag{26}$$

as is seen from equation (24) or the definition (19).

Table 2. Table of parameters for the spheroidal case in terms of $u = 1/2(-1 \pm [1 + 8\varepsilon_{c\perp}\varepsilon_{c\parallel}^{-1}]^{1/2})$ ($\varepsilon_{c\parallel}(\varepsilon_{c\perp})$ is the dielectric function in the direction parallel (perpendicular) to the ellipsoid coating normal) and we have defined $\triangle \equiv [u(u+1)/2]^{1/2} = \sqrt{\varepsilon_{c\perp}\varepsilon_{c\parallel}^{-1}}$.

	$s=1$		$s=0$	
v	x	z	x	z
$\tilde{\alpha}_v$	$-u/2$	$(-u+\triangle)/2$	$(-u+\triangle)/2$	$(1-u)/2$
$\tilde{\beta}$	$(1-u)/2$	$(-u+1+\triangle)/2$	$(-u-\triangle)/2$	$(1-u)/2$
$\tilde{\gamma}$	$1/2-u$	$1/2-u$	$1/2-u$	$1/2-u$
$\tilde{\delta}_v$	0	$(\triangle-1)/2$	0	0
$\tilde{\epsilon}_v$	1	-1	1	-1
$\tilde{\lambda}_v$	1	$-\triangle$	$-\triangle$	0

3.2. Spheroids

In this subsection we present analytical expressions for $H_v(u, s; q)$ and $r_v(u, s; q)$ for *spheriodal* particles (i.e., two of the principal axes are equal). We also express the depolarization factor $n_v(s; q)$ in terms of elementary functions and investigate the cylindrical limit.

Let us consider the solution of Heun's equation for spheroids, i.e., ellipsoids having two of the principal axes equal. We must then distinguish between four cases: (i) prolate spheroids ($a_y = a_z$, or $s = 1$) with the external electric field along x-axis (i.e., along the symmetry axis), (ii) prolate spheroids having the electric field along the z-axis (perpendicular to the symmetry axis), (iii) oblate spheroids ($a_x = a_y$, or $s = 0$) with the external electric field along the x-axis (i.e., perpendicular to the symmetry axis), (iv) oblate spheroids the electric field being along the z-axis (the symmetry axis). Since for spheroids we have $s = 0$ or $s = 1$ the number of singularities in Heun's equation (see equation (15)) is reduced from four to three. Heun's equation is then turned into the class of hypergeometric equations [6, 7], and the solutions can be written in terms of hypergeometric functions. Explicitly in our case

$$H_v(u, s; q) = (1-q)^{\tilde{\delta}_v} F(\tilde{\alpha}_v, \tilde{\beta}_v, \tilde{\gamma}; q) \tag{27}$$

and[4]

$$r_v(u, s; q) = \tilde{\lambda}_v + (u - \tilde{\lambda}_v)\frac{F(\tilde{\alpha}_v, \tilde{\beta}_v + \tilde{\epsilon}_v, \tilde{\gamma}, q)}{F(\tilde{\alpha}_v, \tilde{\beta}_v, \tilde{\gamma}; q)} \tag{28}$$

where the parameters are given in terms of u (for all $v = x, y$ and z, i.e., the field along any of the principal axes) in table 2, and $F(\tilde{\alpha}_v, \tilde{\beta}_v, \tilde{\gamma}; q)$ is the hypergeometric function[5]. Note that (since $F(\tilde{\alpha}_v, \tilde{\beta}_v, \tilde{\gamma}; q = 0) = 1$) we have $r_v(u, s; q) = u$ for a sphere as it should. The hypergeometric functions have well defined analytic continuations to $q = 1$ (see [10, 11] and the cylindrical limit investigation below). The solution for the potential in the coating region as given by equations (6), (13) and (27) is hence valid for the full range of q-values ($0 \leqslant q \leqslant 1$).

[4] In order to obtain equation (28), we have used equations (15.2.3) and (15.2.5) in [10]. The expression for $r_v(u, s; q)$ can be further manipulated by using equation (15.2.14) in the same reference if one wishes.

[5] For large values of $|u|$ the limiting forms of $H_v(u, s; q)$ and $r_v(u, s; q)$ given in subsection 3.3 are useful for practical computations.

For the case of spheroids the depolarization factors (18) can be expressed in terms of elementary functions. By explicitly evaluating the integrals we find

$$\begin{aligned} n_x(s=1;q) &= \frac{(1-q)}{2q^{3/2}}\left[\ln\left(\frac{1+\sqrt{q}}{1-\sqrt{q}}\right)-2\sqrt{q}\right] \\ n_z(s=1;q) &= \frac{1}{4q^{3/2}}\left[2\sqrt{q}-(1-q)\ln\left(\frac{1+\sqrt{q}}{1-\sqrt{q}}\right)\right] \\ n_x(s=0;q) &= \frac{\sqrt{1-q}}{2q^{3/2}}[\arcsin\sqrt{q}-\sqrt{q(1-q)}] \\ n_z(s=0;q) &= \frac{1}{q^{3/2}}[\sqrt{q}-\sqrt{1-q}\arcsin\sqrt{q}]. \end{aligned} \tag{29}$$

We note that the depolarization factors for prolate spheroids satisfy the sum rule $n_x(s = 1; q) + 2n_z(s = 1; q) = 1$ as it should. For oblate spheroids we similarly have the sum rule $2n_x(s = 0; q) + n_z(s = 0; q) = 1$. It is also straight forward to show (by for instance using the integral representation of the hypergeometric functions [10]) that for the isotropic case $r_v(u, s; q)$ as given by equation (28) satisfies equation (20) with the above expressions for the depolarization factors.

We finally consider the cylindrical limit $s = 1$ and $q \to 1$ while $a_x^2 \to \infty$ [1, 2, 9] and field along the short axis (z-axis) as an example of analytic continuation to $q = 1$. We first note that in the cylindrical limit we can use equation (4) to relate the cylindrical radial coordinate $R \equiv \sqrt{y^2+z^2}$ to the variable q according to $R^2 = a_x^2(q^{-1} - 1)$. The potential is given by equations (6), (13) and (27), where the expression for $G_z(u, s; q)$ can be written in terms of $q^{-1} - 1$ (and hence in terms of the radial cylindrical coordinate R) using standard analytical continuations of the hypergeometric functions[6]. We then find that the potential equation (6) in the coating region for a cylinder with an anisotropic coating (field along the z-axis) becomes

$$\Phi|_{\mathrm{cyl}} \propto \Omega_- z R^{\triangle-1} + \Omega_+ z R^{-\triangle-1} \tag{30}$$

with $\Omega_\pm = (a_x)^{1\pm\triangle}\Gamma(1/2-u)\Gamma(\pm\triangle)/(\Gamma([-u\pm\triangle]/2)\Gamma([-u+1\pm\triangle]/2))$ where $\Gamma(z)$ is the gamma function and $\triangle = \sqrt{\varepsilon_{c\perp}\varepsilon_{c\parallel}^{-1}}$ as before ($\triangle = 1$ for an isotropic coating). The same result as given by equation (30) can be obtained by a direct solution of Gauss equation in cylindrical coordinates [4].

3.3. Large $|u|$ expansion and WKB analysis

In this subsection we provide a large $|u|$ (i.e., large coating anisotropy, $|\varepsilon_{c\perp}/\varepsilon_{c\parallel}| \gg 1$) expression for the entities $H_v(u, s; q)$ and $r_v(u, s; q)$, obtained through a direct method as well as using a WKB (Wentzel–Kramer–Brillouin) analysis of Heun's equation.

Let us investigate the functions $H_v(u, s; q)$ and $r_v(u, s; q)$ (see equation (19)) for large $|u|$ ($|u| \gg 1$). The function $H_v(u, s; q)$ satisfies Heun's equation (15). We now make the following *ansatz*:

$$H_v(u, s; q) = \exp\left(-u\int_0^q K_v(q')\,\mathrm{d}q'\right) \tag{31}$$

and note that $H_v(u, s; q = 0) = 1$ provided that $K_v(q)$ is a regular function which is non-singular at $q = 0$. Inserting this ansatz into equation (15) we get the following equation for $K_v(q)$,

$$-uK_v'(q) + u^2K_v^2(q) - u\mathcal{L}_v(q)K_v(q) + \mathcal{M}_v(q) = 0 \tag{32}$$

[6] We here use the analytic continuation of the hypergeometric function as contained in equation (15.3.9) in [10] together with equation (15.3.3) in the same reference.

which is a first-order *nonlinear* equation, which cannot in general be solved. We note that only the first term contains a derivative with respect to q. Retaining only terms to highest (second) order in u the first term drops out and this equation reduces to the *algebraic* equation (i.e., taking the large-u form for $\mathcal{L}_v(q)$ and $\mathcal{M}_v(q)$ and then cancelling a common u^2-factor)

$$K_v^2(q) + \frac{1}{q}K_v(q) + \frac{1}{4q^2}\left(1 - \frac{1}{2f_v}\left[F - \frac{1}{f_v}\right]\right) = 0 \tag{33}$$

with

$$F = F(s;q) \equiv \frac{1}{f_x} + \frac{1}{f_y} + \frac{1}{f_z} \tag{34}$$

where we have assumed $q \neq 1$ and $f_v = f_v(s;q) = 1 - l_v q$ with $l_x = 0$, $l_y = s$ and $l_z = 1$ as before. The solution of this equation is

$$K_v(q) = -\frac{1}{2q}\left\{1 - \left[\frac{1}{2f_v}\left(F - \frac{1}{f_v}\right)\right]^{1/2}\right\} \tag{35}$$

where we have chosen the root that is non-singular at $q = 0$ in order to make the solution regular for $|q| < 1$ and hence $H_v(u, s; q = 0) = 1$ (see equation (31)). For large coating anisotropy ($|u| \gg 1$) the solution of Heun's equation (15) is thus given by equations (31) and (35). These expressions are useful for numerical computations when $|u|$ is large.

Using the result above we can easily obtain an expression for $r_v(u, s; q)$ by noting that $H_v'(u, s; q)/H_v(u, s; q) = -uK_v(q)$. For large $|u|$ ($|u| \gg 1$) equation (19) then becomes

$$r_v(u, s; q) = uR_v(s; q) \qquad R_v(s;q) = \left[\frac{F(s;q)f_v(s;q) - 1}{2}\right]^{1/2}. \tag{36}$$

Note that $r_v(u, s; q)$ is proportional to u for large $|u|$ multiplied by a geometric shape function $R_v(s; q)$. The results obtained above should prove useful for instance for the electric response at frequencies where the coating dielectric function has a resonance.

The results above can also be obtained through a WKB (or phase integral) method; for a detailed WKB analysis of Heun's equation, see [12]. We make the following ansatz for the solution of Heun's equation (15):

$$H_v(u, s; q) = \Psi_v(q)\exp\left[-\frac{1}{2}\int^q \mathcal{L}_v(q')\,\mathrm{d}q'\right]. \tag{37}$$

One then obtains the equation for $\Psi_v(q)$ as

$$\Psi_v''(q) - \mathcal{T}_v(q)\Psi_v(q) = 0 \tag{38}$$

where $\mathcal{T}_v(q) = \mathcal{L}_v'(q)/2 + \mathcal{L}_v^2(q)\big/4 - \mathcal{M}_v(q)$ with $\mathcal{L}_v(q)$ and $\mathcal{M}_v(q)$ being given in equation (15). Note that the equation above is an exact reformulation of Heun's equation (15). Equation (38) has the same form as the one-dimensional Schrödinger equation for a particle moving in a potential [13]. Methods developed in this field of physics can thus be directly applied to the present problem. We here limit the discussion to the zeroth order WKB analysis of equation (38) [13], which gives the acceptable solution

$$\Psi_v(q) = \exp\left[-\int^q \mathcal{T}_v(q')^{\frac{1}{2}}\,\mathrm{d}q'\right]. \tag{39}$$

For large $|u|$ the solution of Heun's equation as given by equations (38) and (39) agrees with the result given by equations (31) and (35). Since we are concerned with the logarithmic derivative of $H_v(u, s; q)$ through $r_v(u, s; q)$, higher order WKB yields insignificant correction to equation (36) for $|u| \gg 1$.

4. Matching boundary conditions

In this section we match the solutions for the potentials in the inside, coating and surrounding in order to obtain the electric fields in the different regions as well as the polarizability for an ellipsoidal particle with an anisotropic coating.

Using the results from the previous two sections together with the appropriate physical constraints we now give the potential in the inner ellipsoid, the coating and the medium outside the ellipsoid respectively. We have the following solutions for the potentials of the inner ellipsoid and the coating ($v = x, y, z$):

$$\Phi_{\text{in}} = A_v v G_v(u = 1, s; q) = A_v v \tag{40}$$

and

$$\Phi_{\text{coat}} = B_v v G_v(u_-, s; q) + C_v v G_v(u_+, s; q) \tag{41}$$

where $G_v(u, s; q)$ is given by equation (13) and we have required the potential to be non-singular. For the potential outside the coated ellipsoid we write

$$\Phi_{\text{out}} = \Phi_{0v} + \Phi_p \tag{42}$$

where

$$\Phi_p = D_v v G_v(u = -2, s; q)|_{q \neq 1}. \tag{43}$$

We have required that the potential satisfies $\Phi \to \Phi_{0v}$ (see equation (2)) at infinity, which in turn requires us to assume $q \neq 1$ (for $q = 1$ the induced potential Φ_p as given by equation (43) is not zero at infinity as is seen, for instance, in the cylindrical result (30) with $\triangle = 1$). The subsequent results obtained in this section are therefore only valid for $0 \leqslant q < 1$ and this is also the reason why we required that $q \neq 1$ in the definition of $r_v(u, s; q)$ in equation (19) (for $q = 1$ we must replace Φ_p in equation (43) by the appropriate linear combination of the $u_+ = 1$ and $u_- = 2$ isotropic solutions that vanish at infinity). The unknown constants A_v, B_v, C_v and D_v are determined through the boundary conditions.

Having the formal solutions for the potential we now proceed to match the solutions across the boundaries.We impose the conditions that the potential Φ and normal component of the displacement field are continuous, so that $\varepsilon_\parallel \partial\Phi/\partial\xi$ is continuous, to determine the unknown constants A_v, B_v, C_v and D_v. In order to determine the normal component of the displacement field we have the useful result $\partial v/\partial q = -v/(2qf_v)$. Furthermore we find that

$$\frac{\partial}{\partial q}[v G_v(u, s; q)] = -\frac{v}{2qf_v} q^{(1-u)/2} H_v(u, s; q) r_v(u, s; q) \tag{44}$$

where $r_v(u, s; q)$ is given in equation (19) and $f_v = f_v(s; q)$ has been defined previously. The entity $r_v(u, s; q)$ enters the analysis through the condition that the displacement field is continuous. If we also define $\varepsilon_i \equiv \varepsilon_{\text{in}}/\varepsilon_{\text{m}}$, $\varepsilon_\parallel \equiv \varepsilon_{\text{c}\parallel}/\varepsilon_{\text{m}}$ and $\varepsilon_\perp \equiv \varepsilon_{\text{c}\perp}/\varepsilon_{\text{m}}$ and recall that the inner and outer ellipsoidal surfaces correspond to $q = e_i^2$ ($\xi = 0$) and $q = e_o^2$ ($\xi = t$), the unknown constants are found to be

$$A_v = E_0 \frac{H_v^i(u_+)}{Q_v H_v^o(u_+)} \left(\frac{e_o}{e_i}\right)^{(u_+-1)} \varepsilon_\parallel \left[r_v^i(u_+) - r_v^i(u_-)\right]$$

$$B_v = E_0 \frac{e_o^{(u_--1)}}{Q_v n_v^o H_v^o(u_-)} \rho_v \left[\varepsilon_\parallel r_v^i(u_+) - \varepsilon_i\right]$$

$$C_v = -E_0 \frac{e_o^{(u_+-1)}}{Q_v n_v^o H_v^o(u_+)} \left[\varepsilon_\parallel r_v^i(u_-) - \varepsilon_i\right]$$

$$D_v = E_0 \frac{1}{3Q_v n_v^o} \frac{\left(f_x^o f_y^o f_z^o\right)^{1/2}}{e_o^3} \left\{\left[\varepsilon_\parallel r_v^o(u_+) - 1\right]\left[\varepsilon_\parallel r_v^i(u_-) - \varepsilon_i\right] - \rho_v\left[\varepsilon_\parallel r_v^o(u_-) - 1\right]\left[\varepsilon_\parallel r_v^i(u_+) - \varepsilon_i\right]\right\} \quad (45)$$

where[7]

$$\rho_v \equiv \left(\frac{e_o}{e_i}\right)^{(u_+-u_-)} \frac{H_v^o(u_-)H_v^i(u_+)}{H_v^o(u_+)H_v^i(u_-)} \quad (46)$$

and

$$Q_v = \left[\varepsilon_\parallel r_v^o(u_+) + 1/n_v^o - 1\right]\left[\varepsilon_\parallel r_v^i(u_-) - \varepsilon_i\right] - \rho_v\left[\varepsilon_\parallel r_v^o(u_-) + 1/n_v^o - 1\right]\left[\varepsilon_\parallel r_v^i(u_+) - \varepsilon_i\right]. \quad (47)$$

We have introduced the short-hand notation: $r_v^i(u_\pm) \equiv r_v\left(u_\pm, s, e_i^2\right)$ and $r_v^o(u_\pm) \equiv r_v\left(u_\pm, s, e_o^2\right)$. We have also defined $H_v^o(u_\pm) \equiv H_v\left(u_\pm, s; e_o^2\right)$ and $H_v^i(u_\pm) \equiv H_v\left(u_\pm, s; e_i^2\right)$ and $f_v^o \equiv f\left(s; e_o^2\right)$. It is interesting to note that it is only the depolarization factor $n_v^o \equiv n_v\left(s; e_o^2\right)$ of the *outer* surface that appears explicitly in the expressions above.

The electric fields are obtained by taking the gradient of the potential, explicitly $\vec{E} = -\vec{\nabla}\Phi$. Let us for completeness give the results for the electric fields in the three different regions. The field $\vec{E}_{\text{in}} = -\vec{\nabla}\Phi_{\text{in}}$ inside the ellipsoid is particularly simple. When the field is along the v-axis we have

$$\vec{E}_{\text{in}} = -A_v\hat{v} \quad (48)$$

where $\hat{v}$ is a unit vector in the v-direction and A_v is given in equation (45). The electric field inside the ellipsoid is thus constant along the direction of the external field. We note for instance that the electric field is zero inside the ellipsoid if $\varepsilon_\parallel = 0$. This occurs if $\varepsilon_\parallel^{-1}$ has a resonance, then the entire potential drop is across the coating and there is no potential drop 'left' for the inner part. The electric field in the coating region is $\vec{E}_{\text{coat}} = -\vec{\nabla}\Phi_{\text{coat}}$, which explicitly becomes

$$\vec{E}_{\text{coat}} = -\Bigg\{[B_v H_v(u_-, s; q)q^{(1-u_-)/2} + C_v H_v(u_+, s; q)q^{(1-u_+)/2}]\hat{v} + \frac{v/f_v}{\left(x^2/f_x^2 + y^2/f_y^2 + z^2/f_z^2\right)^{1/2}}[B_v H_v(u_-, s; q)(r_v(u_-, s; q) - 1)q^{(1-u_-)/2} + C_v H_v(u_+, s; q)(r_v(u_+, s; q) - 1)q^{(1-u_+)/2}]\hat{\xi}\Bigg\} \quad (49)$$

where $\hat{\xi}$ is a unit vector perpendicular to the ellipsoidal surface ξ = constant as previously and x, y and z are the usual Cartesian coordinates. The electric field in the coating thus in

[7] For two important cases ρ_v (see equation (46)) can be expressed in terms of $r_v^o(u_\pm)$: (i) when $|u|\bar{\delta} \ll 1$, i.e., for a small relative coating thickness and not 'too large' u. (ii) We have a thin coating, $\bar{\delta} \ll 1$ and large $|u|(|u| \gg 1)$. For both of these cases we have $\rho_v \approx \exp\{-\bar{\delta}[r_v^o(u_+) - r_v^o(u_-)]/(2f_v^o)\}$. This result can be considered as a thin relative coating ($\bar{\delta} = t/b_x^2 \ll 1$) result which is valid for most practical purposes.

general has components in both the $\hat{v}$-direction and the $\hat{\xi}$-direction. Finally the electric field $\vec{E}_{\text{out}} = -\vec{\nabla}\Phi_{\text{out}}$ outside the ellipsoid is

$$\vec{E}_{\text{out}} = E_0\hat{v} - 3D_v(f_x f_y f_z)^{-1/2} n_v(s;q) q^{3/2} \times \left[\hat{v} - \frac{v/f_v}{\left(x^2/f_x^2 + y^2/f_y^2 + z^2/f_z^2\right)^{1/2}} \frac{1}{n_v(s;q)} \hat{\xi} \right]. \tag{50}$$

The three expressions above give the complete result for the electric field in all space. Using equations (48)–(50) together with the explicit expressions for $H_v(u, s; q)$ and $r_v(u, s; q)$ from the previous section, the electric field distribution in and around an ellipsoid with an anisotropic coating can thus be conveniently obtained.

Let us finally obtain the polarizability α_{vv} ($v = x, y, z$) of the ellipsoid. By taking the asymptotic limit of Φ_p (see equations (43) and (45)) we identify the induced dipole moment as $p_v = 4\pi\varepsilon_0\varepsilon_{\text{m}}\alpha_{vv}E_0$ (ε_0 is the permittivity of free space) where we have introduced the polarizability of an ellipsoid with an anisotropic coating according to

$$\alpha_{vv} = \frac{V_o}{4\pi n_v^o} \times \frac{\left[\varepsilon_\parallel r_v^o(u_+) - 1\right]\left[\varepsilon_\parallel r_v^i(u_-) - \varepsilon_i\right] - \rho_v\left[\varepsilon_\parallel r_v^o(u_-) - 1\right]\left[\varepsilon_\parallel r_v^i(u_+) - \varepsilon_i\right]}{\left[\varepsilon_\parallel r_v^o(u_+) + 1/n_v^o - 1\right]\left[\varepsilon_\parallel r_v^i(u_-) - \varepsilon_i\right] - \rho_v\left[\varepsilon_\parallel r_v^o(u_-) + 1/n_v^o - 1\right]\left[\varepsilon_\parallel r_v^i(u_+) - \varepsilon_i\right]} \tag{51}$$

with $V_o = 4\pi b_x b_y b_z/3$ being the (outer) volume of the ellipsoid. This result for the polarizability, equation (51), of an ellipsoidal particle with an anisotropic coating is the main result of this study. The geometry of the particle enters through the five geometric entities n_v^o, $r_v^i(u_\pm)$ and $r_v^o(u_\pm)$. The standard isotropic depolarization factor n_v^o depends only on the shape (e_o^2 and s), whereas $r_v^i(u_-)$ and $r_v^o(u_+)$ couples the geometry to $\varepsilon_\perp\varepsilon_\parallel^{-1}$. Let us now consider some limits. We start with the *isotropic* ellipsoidal shell ($\varepsilon_\parallel = \varepsilon_\perp$). We then have $u_+ \to 1$ and $u_- \to -2$ and $r_v(u, s; q)$ is then given by equation (20). We also have $\rho_v = V_i n_v^o / V_o n_v^i$ where $V_i = 4\pi a_x a_y a_z/3$ is the volume of the inner ellipsoid and $n_v^i \equiv n_v\left(s; e_i^2\right)$ is the depolarization factor for the inner surface. The corresponding result for the polarizability then agrees with that found in [2] as it should. In the *spherical* limit we have $n_v^o = 1/3, r_v^{i,o}(u) \to u$ and $\rho = (a/b)^{(u_+ - u_-)}$ where $a = a_x = a_y = a_z$ is the radius of the inner spherical surface, whereas $b = b_x = b_y = b_z$ is the radius of the outer surface. This result agrees with the result obtained in [3]. For the *cylindrical* limit the results above for the polarizability are not defined (see the discussion following equation (43)). However using the appropriate solutions (see equation (30)) that satisfy the physical constraints and matching the solutions across the boundaries the expression for the polarizability can be worked out separately [4]. One then finds that the polarizability (field along short axis, i.e., the z-axis) takes the same form as given by equation (51), with: $n_z^o = 1/2, r_z^i(u_\pm) = r_z^o(u_\pm) = \pm\Delta = \pm\sqrt{\varepsilon_\perp/\varepsilon_\parallel}$ and $\rho_z = (a/b)^{2\Delta}$, where $a = a_y = a_z$ is the radius of the inner cylindrical surface and $b = b_y = b_z$ is the outer cylindrical radius. For the general case the relevant expressions are given in subsection 3.1. For spheroidal particles the results in subsection 3.2 should be used. Subsection 3.3 contains the results necessary to obtain the polarizability for large $|u|$.

We finally point out that the results above are directly applicable to the case of a permeable ellipsoid with an anisotropic coating in a constant external *magnetic* field B_0 [5]. The induced magnetic dipole moment $\vec{m}$ is then related to the external magnetic field according to $m_v = 4\pi(\mu_0\mu_{\text{m}})^{-1}\alpha_{vv}B_0$ ($v = x, y$ or z) where μ_{m} is the magnetic permeability of the medium surrounding the particle (μ_0 is the permeability of free space). The magnetic polarizability α_{vv} is given by equation (51) with the replacements $\varepsilon_i \to \mu_{\rm in}/\mu_{\rm m}$, $\varepsilon_\parallel \to \mu_{\rm c\parallel}/\mu_{\rm m}$ and $\varepsilon_\perp \to \mu_{\rm c\perp}/\mu_{\rm m}$, where $\mu_{\rm in}$ is the magnetic permeability of the inner part of the ellipsoid and $\mu_{\rm c\perp}(\mu_{\rm c\parallel})$ is the permeability of the coating in the direction perpendicular (parallel) to the coating normal.

5. Summary and conclusions

In this paper we have studied the electric and magnetic response of an ellipsoidal particle with a dielectrically anisotropic coating (the dielectric function of the coating being different parallel and perpendicular to the coating normal). For the coating region we found that the potential can be written in terms of solutions to Heun's equation (15), which was derived from the three-dimensional Gauss equation for the potential in the ellipsoidal coordinates. We gave solutions to Heun's equation in three forms: for the general case we obtained solutions in terms of a series expansion. For the case of spheroidal particles we obtained solutions in terms of hypergeometric functions. When the coating anisotropy is large (i.e., $|\varepsilon_\perp/\varepsilon_\parallel| \gg 1$, where $\varepsilon_\perp$ ($\varepsilon_\parallel$) is the relative dielectric function perpendicular (parallel) to the coating normal) we derived a simple asymptotic form of the solution for the potential. By matching the solutions across the boundaries we obtained the ellipsoidal particle polarizability, equation (51), in terms of the standard depolarization factors and logarithmic derivatives of the Heun equation solutions.

From the mathematical point of view the present study provides two new insights into Heun's equation. (i) In general no integral representation for the solution has been found. However, the result as contained in equation (17) provides such a representation although for a very specific set of parameter values. (ii) Whenever the parameters in Heun's equation are parametrized by a parameter denoted by, say, u then the techniques used in subsection 3.3 to convert Heun's equation into an algebraic equation can prove useful in order to obtain the large $|u|(|u| \gg 1)$ form for the solution. We also want to point out that the theoretical analysis of Heun's equation is not as developed as is the analysis of, for instance, the hypergeometric equation. Progress in this field of mathematics may thus provide further insights into the present problem.

The results obtained here for the electric and magnetic response of an ellipsoid with an anisotropic coating are general and could find applications in many areas of physics. For example, the simplest and widest application of technological importance (e.g., in environmental studies, detectors, etc) would be the scattering, absorption and emission of light by ellipsoidal nanoparticles with ultra-thin coating (e.g., gold film on a nanoparticle), where the anisotropy is naturally incorporated in ultra-thin films due to growth induced anisotropy; a detailed theoretical study here would involve numerical computations, along the lines described by Mishchenko *et al* in [9] with only nominal modifications of the computer programs given by them. Such theoretical modelling would help in selecting a suitable coating for tailor made optical properties. We are currently working on some specific cases for future publications. Another possible application is to the physics of fullerenes. It is known that fullerenes are non-spherical [14], and their dielectric functions are generally anisotropic [3], which affect their physical (e.g., optical, interparticle forces, etc) properties. The results obtained here can be directly applied to the response of fullerenes with suitable modelling. The most interesting application of the result of this study is the electric [15], magnetic [16] and electromagnetic response of biological cells. We are also working on this problem by incorporating microscopic models for the cell membrane as well as for the cell interior into the present scheme and shall report elsewhere.

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